Chap 2. Relation and Function

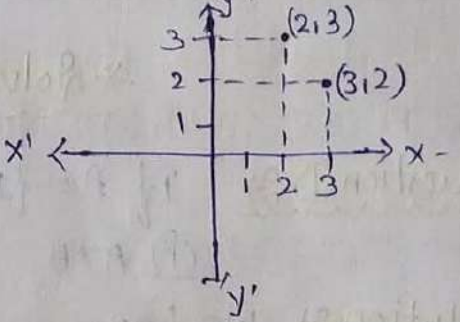
Oredered pair :→ Two numbers a and b listed in a specific order and enclosed in parentheses form an ordered pair (a,b).

→ In the ordered pair (a,b), we call a as first component or member and b as second component or member.

→ By interchanging the positions of the components, the ordered pair is changed.

Thus, $(a,b) \neq (b,a)$

For ex:- $(2,3)$ and $(3,2)$ are not same and represented as



Equality of two ordered pairs :→ Two ordered pairs (a,b) and (c,d) will be equal if

$$\boxed{(a,b) = (c,d) \Rightarrow a = c \text{ and } b = d}$$

Example:- Find a and b, when $(a-1, b+5) = (2,3)$

Solution :→ $(a-1, b+5) = (2,3)$

Here $a-1 = 2$

$\boxed{a = 3}$ Ans

and $b+5 = 3$

$\boxed{b = -2}$ Ans.

Cartesian product of two sets :→ Let A and B be two nonempty sets. Then, the Cartesian product of A and B is the set denoted by $A \times B$, consisting of all ordered pairs (a,b) such that $a \in A$ and $b \in B$.

$$\therefore \quad A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$$

If $A = \phi$ or $B = \phi$, we define $A \times B = \phi$.

Note :→ (i) If $n(A) = p$ and $n(B) = q$ then $n(A \times B) = pq$ and $n(B \times A) = pq$.

(ii) If at least one of A and B is infinite the $A \times B$ and $B \times A$ is infinite.

* Solved Sums :→ *

Question ① :→ If $A = \{1,3,5\}$ and $B = \{2,3\}$ then find :

(i) $A \times B$ (ii) $B \times A$ (iii) $(A \times B) \cap (B \times A)$

Solution :→ We have

$$A = \{1,3,5\}, \quad B = \{2,3\}$$

(i) $A \times B = \{1,3,5\} \times \{2,3\}$

$= \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3)\}$

(ii) $B \times A = \{2,3\} \times \{1,3,5\}$

$= \{(2,1), (2,3), (2,5), (3,1), (3,3), (3,5)\}$

(iii) $(A \times B) \cap (B \times A) = \{(3,3)\}$.

Question ② :→ If $A = \{1,2,3\}$, $B = \{3,4\}$ and $C = \{4,5,6\}$ the find:

(i) $A \times (B \cap C)$ (ii) $(A \times B) \cap (A \times C)$

(iii) $A \times (B \cup C)$ (iv) $(A \times B) \cup (A \times C)$

Solution :→ given that $A = \{1,2,3\}$, $B = \{3,4\}$ and $C = \{4,5,6\}$

$\therefore$ (i) $A \times (B \cap C)$

$\Rightarrow \{1,2,3\} \times \{4\}$

$\Rightarrow \{(1,4), (2,4), (3,4)\}$ Ans.

(ii) $(A\times B)\cap(A\times C)$

$\Rightarrow A\times B=\{1,2,3\}\times\{3,4\}$

$=\{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}$

$\Rightarrow A\times C=\{1,2,3\}\times\{4,5,6\}$

$=\{(1,5),(1,4),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5)(3,6)\}$

$\therefore (A\times B)\cap(A\times C)=\{(1,4),(2,4),(3,4)\}$ Ans.

(iii) $A\times(B\cup C)$

$=\{1,2,3\}\times\{3,4,5,6\}$

$=\{(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5)(2,6)(3,3)(3,4)(3,5),(3,6)\}$

(iv) $(A\times B)\cup(A\times C)$

$=\{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}\cup(1,5),(1,4),(1,6),(2,4),(2,5),(2,6),(3,4),$
$(3,5),(3,6)\}$

$=\{(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,3),(3,4),(3,5),(3,6)\}$ Ans.

Question ③ :- Let $A=\{x\in N: x^2-5x+6=0\}$, $B=\{x\in W: 0\le x<2\}$ and $C=\{x\in N: x<3\}$. Verify that

Solution :- We have

$A=\{x\in N: x^2-5x+6=0\}$

$=\{x\in N: (x-2)(x-3)=0\}$

$\therefore A=\{2,3\}$

$\therefore B=\{1,2\}$

$\therefore C=\{1,2\}$

$\therefore$ (i) $A\times(B\cup C)=(A\times B)\cup(A\times C)$

L.H.S. $A\times(B\cup C)$

$\Rightarrow \{2,3\}\times\{1,2\}$

$\Rightarrow\{(2,1),(2,2),(3,1),(3,2)\}$

R.H.S. $(A\times B)\cup(A\times C)$

$\Rightarrow\{(2,1)(2,2),(3,1)(3,2)\}\cup\{(2,1)(2,2),(3,1)(3,2)\}$

$\Rightarrow\{(2,1),(2,2),(3,1),(3,2)\}$

Hence $A\times(B\cup C) = A\times B\cup A\times C$ Proved

(ii) $A\times(B\cap C)=(A\times B)\cap(A\times C)$

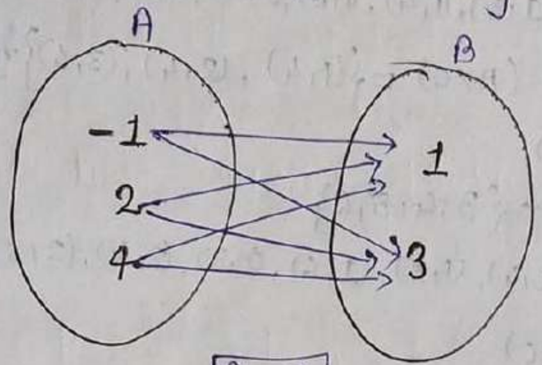
prove yourself.

Arrow diagram of $A \times B$:- Let $A = \{-1, 2, 4\}$ and $B = \{1, 3\}$

Then,

$A \times B = \{(-1,1), (-1,3), (2,1), (2,3), (4,1), (4,3)\}$

Then $A \times B$ can be represented by Arrow diagram.



$\boxed{A \times B}$

Order Triplet :- Three numbers a, b and c listed in a specific order and enclosed in parenthesis form an ordered ~~pair~~ triplet (a, b, c).

Thus, $(1,2,3) \neq (2,1,3) \neq (3,2,1)$, etc.

For any non-empty set A, we define

$$\boxed{A \times A \times A = \{(a,b,c) : a, b, c \in A\}}$$

Example :- If $A = \{1,2\}$, find $A \times A \times A$

Solution :- $A \times A = \{1,2\} \times \{1,2\}$

$= \{(1,1), (1,2), (2,1), (2,2)\}$

$\therefore A \times A \times A = (A \times A) \times A$

$= \{(1,1), (1,2), (2,1), (2,2)\} \times \{1,2\}$

$= \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,2), (2,1,1), (2,2,1), (2,2,2)\}$

Ans

ASSIGNMENT ①

1. Find the values of a and b, when:
(i) $(a+3, b-2) = (5, 1)$ (ii) $(a+b, 2b-3) = (4, -5)$
(iii) $\left(\frac{a}{3}+1, b-\frac{1}{3}\right) = \left(\frac{5}{3}, \frac{2}{3}\right)$ (iv) $(a-2, 2b+1) = (b-1, a+2)$

2. If $A = \{0, 1\}$ and $B = \{1, 2, 3\}$, show that $A \times B \neq B \times A$

3. If $P = \{a, b\}$ and $Q = \{x, y, z\}$, show that $P \times Q \neq Q \times P$.

4. If $A = \{2, 3, 5\}$ and $B = \{5, 7\}$, find: $A \times B$, $B \times A$, $A \times A$, $B \times B$.

5. If $A = \{x \in N : x \leq 3\}$ and $B = \{x \in W, x < 2\}$, find $A \times B$ and $B \times A$. Is $A \times B = B \times A$?

6. If $A = \{1, 3, 5\}$, $B = \{3, 4\}$ and $C = \{2, 3\}$, verify that
(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

7. Let $A = \{x \in W : x < 2\}$, $B = \{x \in N : 1 \leq x \leq 4\}$ and $C = \{3, 5\}$. Verify that (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

8. If $A \times B = \{(-2, 3), (-2, 4), (0, 3), (0, 4), (3, 3), (3, 4)\}$, find A and B.

9. Let $A = \{2, 3\}$ and $B = \{4, 5\}$. Find $A \times B$. How many subsets will $A \times B$ have?

10. Let $A \times B = \{(a, b) : b = 3a - 2\}$. If $(x, -5)$ and $(2, y)$ belong to $A \times B$, find the values of x and y

11. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $a \neq b \neq c$ and $(a, 0), (b, 1), (c, 0)$ are in $A \times B$, find A and B.

12. Let $A = \{-2, 2\}$ and $B = \{0, 3, 5\}$. Find:
(i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

13. If $A = \{5, 7\}$, find (i) $A \times A$ and (ii) $A \times A \times A$

14). Let $A = \{-3, -1\}$, $B = \{1, 3\}$ and $C = \{3, 5\}$. Find
(i) $A \times B$ (ii) $(A \times B) \times C$ (iii) $B \times C$ (iv) $A \times (B \times C)$

** Relations **

Relation: Let A and B be non-empty sets. Then a relation R from A to B is a subset of $A \times B$.

Thus, R is relation from $A \times B \Leftrightarrow \boxed{R \subseteq A \times B}$

→ If $(a, b) \in R$ then we say that 'a is related to b' and we we write aRb.

→ If $(a, b) \notin R$ then 'a is not related to b' and we write $a \not R b$.

Domain, Range and codomain of a relation:

Let R be a relation from A to B. Then $R \subseteq (A \times B)$.

(i) The set of all first coordinates of elements of R is called the domain of R, written as $dom(R)$.

(ii) The set of all second coordinates of elements of R is called the range of R, denoted by $range(R)$.

(iii) The set B is called the codomain of R.

$Dom(R) = \{a : (a, b) \in R\}$ and $Range(R) = \{b : (a, b) \in R\}$

Total number of relations from A to B:

Let $n(A) = p$ and $n(B) = q$. Then

$n(A \times B) = pq$.

We know that every subset of $A \times B$ is a relation from A to B

→ $\boxed{\text{Total no. of Subsets of } A \times B \text{ is } 2^{pq}}$

∴ Total number of relations from A to $B = 2^{pq}$

* Representation of a relation :→ *

Let A and B be two given sets. Then, a relation $R \subseteq A \times B$ can be represented in any of the forms given below.

(i) Roster form :→ In this form, R is given as a set of ordered pairs.

Example(1) :→ Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 4, 9\}$

Let $R = \{(-2,4), (-1,1), (0,0), (1,1), (2,4)\}$

(i) Show that R is a relation from A to B

(ii) find dom(R), range(R) and codomain of R.

Solution :→ (i) Since $R \subset A \times B$, so R is a relation from A to B

Note that $-2R4$, $-1R1$, $0R0$ etc.

(ii) Dom(R) = Set of first coordinates of elements of R

$= \{-2, -1, 0, 1, 2\}$ Ans

Range (R) = Set of second coordinates of elements of R

$= \{0, 1, 4\}$

co-domain of $R = \{0, 1, 4, 9\} = B$.

(ii) Set Builder form :→ Under this method, for every $(a,b) \in R$, a general relation is being given between 'a' and 'b'

Using this relation, all the elements of R can be obtained.

Example(2) :- Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 6, 9\}$

Define a relation A to B given by

$R = \{(a,b) : a \in A, b \in B \text{ and } (a-b) \text{ is odd}\}$

$\therefore$ In roster form $R = \{(1,4), (1,6), (2,9), (3,4), (3,6), (5,4), (5,6)\}$.

Question: Let $R = \{(x, x^3) : x \text{ is a prime no. less than } 10\}$

(i) Write R in Roster form

(ii) Find dom(R) and range(R).

Solution: prime numbers less than 10 are 2, 3, 5, 7

(i) $R = \{(2, 2^3), (3, 3^3), (5, 5^3), (7, 7^3)\}$

$\therefore R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

(ii) $Dom(R) = \{2, 3, 5, 7\}$

$Range(R) = \{8, 27, 125, 343\}$. Ans

Question: Let $R = \{(x, y) : x \text{ and } y \text{ are integers and } xy = 4\}$

(i) Write R in Roster form

(ii) find dom(R) and range(R)

Solution: clearly we have

(i) $R = \{(-4, -1), (-2, -2), (-1, -4), (1, 4), (2, 2), (4, 1)\}$

(ii) $Dom(R) = \{-4, -2, -1, 1, 2, 4\}$

$Range(R) = \{-1, -2, -4, 1, 2, 4\}$

(iii) Arrow Diagram: Let R be a relation from A to B. first, we draw two bounded figures to represent A and B respectively.

Example (5): Let $A = \{1, 2, 3, 4, 5\}$. Define a relation R from A to A by $R = \{(x, y) : y = 2x - 3\}$

(i) Depict R using an arrow diagram.

(ii) find dom(R) and range(R).

Solution:

$x = 1 \Rightarrow y = 2 - 3 = \boxed{-1} \notin A$

$x = 2 \Rightarrow y = 4 - 3 = \boxed{1}$

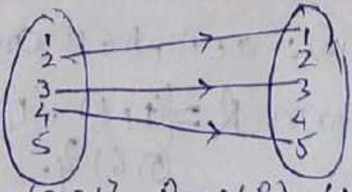
$x = 3 \Rightarrow y = 6 - 3 = \boxed{3}$

$x = 4 \Rightarrow y = 8 - 3 = \boxed{5}$

$x = 5 \Rightarrow y = 10 - 3 = \boxed{7} \notin A$

$\therefore R = \{(2, 1), (3, 3), (4, 5)\}$.

(i)



(ii) $dom(R) = \{2, 3, 4\}$, $Range(R) = \{1, 3, 5\}$ Ans

Inverse relation :→ Let R be a binary relation on a set A. Then, the inverse of R, denoted by R^{-1} is a binary relation on A defined by $R^{-1} = \{(b,a) : (a,b) \in R\}$

Clearly $(a,b) \in R \Leftrightarrow (b,a) \in R^{-1}$

Also, $dom(R) = range(R^{-1})$ and $range(R) = dom(R^{-1})$

Question :→ Let A be a set of first ten natural numbers. Let R be a binary relation on A defined by
$R = \{(a,b) : a, b \in A$ and $a+2b = 10\}$
Express R and R^{-1}

Solution :→

$a + 2b = 10 \Rightarrow b = \frac{10-a}{2}$

Now $(a = 2 \Rightarrow b = 4), (a = 4, \Rightarrow b = 3)$
$(a = 6 \Rightarrow b = 2), (a = 8 \Rightarrow b = 1)$

$\therefore R = \{(2,4), (4,3), (6,2), (8,1)\}$

$\Rightarrow R^{-1} = \{(4,2), (3,4), (2,6), (1,8)\}$

(i) $dom(R) = \{2,4,6,8\} = rang(R^{-1})$

(ii) $range(R) = \{4,3,2,1\}, Dom(R) = \{4,3,2,1\}$.

Binary Relation :→ Let A be a nonempty set. Then, every subset of $A \times A$ is called binary relation or simply a relation on A.

Question :→ Let $A = \{1,2,3\}$ and $R = \{(1,2), (2,2), (3,1), (3,2)\}$. Show that R is a binary relation on A. Find its domain and range.

Solution :→ $R = \{(1,2), (2,2), (3,1), (3,2)\}$

clearly $R \subset A \times A$ and so R is a relation on A.

$Dom(R) = \{1,2,3\}$

$Range(R) = \{1,2\}$ Ans

(iii) Now let $A = \{2,3,4\}$

$\therefore f(A) = \{4,9,16\}$ Ans

Question②: let $X = \{2,3,4,5\}$ and $Y = \{7,9,11,13,15,17\}$

Define a relation f from X to Y by:

$f = \{(x,y) : x \in X, y \in Y \text{ and } y = 2x+3\}$

(i) Write f in roster form

(ii) find dom(f) and range(f).

(iii) Show that f is a function from X to Y.

Solution: given that

$X = \{2,3,4,5\}$

$y = 2x+3$

(i) $\therefore f = \{(2,7),(3,9),(4,11),(5,13)\}$

(ii) clearly, domain(f) $= \{2,3,4,5\}$

range(f) $= \{7,9,11,13\}$.

(iii) It is clear that no two distinct ordered pairs in f have the same first coordinate.

$\therefore f$ is a function from X to Y. Ans.

Question③: Which of the following relations are functions? Give reasons. In case of a function, find its domain and range.

(i) $f = \{(1,3),(1,5),(2,3),(2,5)\}$

(ii) $f = \{(2,1),(5,1),(8,1),(11,1)\}$

(iii) $h = \{(2,1),(4,2),(6,3),(8,4),(10,5),(12,6)\}$.

Solution: (i) $f = \{(1,3),(1,5),(2,3),(2,5)\}$

Here, one element, namely 1 has two images 3 and 5 under f.

$\therefore f$ is not a function.

(ii) $g = \{(2,1),(5,1),(8,1),(11,1)\}$

clearly, no two distinct ordered pairs in g have the same first coordinate. So, g is a function.

(iii) $h = \{(2,1)(4,2),(6,3),(8,4),(10,5),(12,6)\}$

clearly, no two distinct ordered pairs in h have the same first coordinate, so h is a function.

Equal functions :→ Two functions f and g are said to be equal if

(i) $dom(f) = dom(g)$

(ii) co-domain of (f) = co-domain of (g)

(iii) $f(x) = g(x)$ for every x in their common domain.

Question ⑤ :→ Let $A = \{1, 2\}$ and $B = \{3, 6\}$ and f and g be functions from A to B, defined by $f(x) = 3x$ and $g(x) = x^2 + 2$. Show that $f = g$.

Solution :→ Clearly, we have

$dom\,f = dom\,g = \{1, 2\}$.

$codomain(f) = codomain(g) = \{3, 6\}$

Also, $f(1) = 3$, $g(1) = 3$

$f(2) = 6$ $\quad g(2) = 6$

Thus, $\boxed{f(x) = g(x)}$ for all $x \in A$

Hence f and g are equal functions.

Question ⑥ :→ Let $f : Z \to Z : f(x) = x^2$ and $g : Z \to Z : g(x) = |x|^2$ for all $x \in Z$. Show that $f = g$

Solution :→ We have

$dom(f) = dom(g) = Z$

and $codomain(f) = codomain(g) = Z$

Also, for all $x \in Z$, we have

$f(x) = x^2$ and $g(x) = |x|^2 = x^2$

$\therefore$ $f(x) = g(x)$ for all $x \in Z$

Hence f and g are equal functions.

Question 7 :-> Let $f: R \to R : f(x) = (x+2)$ and $g: R-\{2\} \to R$
$g(x) = \frac{x^2-4}{x-2}$. Show that $f \neq g$. Redefine f and g s.t. $f = g$.

Solution :-> We have, $dom(f) = R$
$dom(g) = R-\{2\}$
$\therefore dom(f) \neq dom(g)$
So, we have $f \neq g$. proved.
For every real no. $x \neq 2$, we have

$$g(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{(x-2)} = (x+2)$$

Thus $f(x) = g(x)$ for all $x \in R-\{2\}$

$\therefore f = g$ only when, they are redefined as under:
$f: R-\{2\} \to R : f(x) = (x+2)$ and
$g: R-\{2\} \to R : g(x) = \frac{x^2-4}{x-2}$.

Question 8 :-> Let $f = \{(-1,-3), (0,-1), (1,1), (2,3)\}$ be a function, described by the formula, $f(x) = \alpha x + \beta$. Then, find the values of α and β. Also find the formula.

Solution :-> Here $f(x) = \alpha x + \beta$ —— ①
Also, $f(-1) = -3$, $f(0) = -1$, $f(1) = 1$
$f(2) = 3$
$\therefore$ we get from eqn ①
$-3 = -\alpha + \beta$
$\Rightarrow \alpha - \beta = 3$ —— (ii)
and $\boxed{-1 = \beta}$ —— (iii)
$\therefore \boxed{\alpha = 2}$ $\beta = -1$
Hence $\boxed{f(x) = 2x - 1}$ is the required formula.

Question 9:- Let $f: R \to R : f(x) = x^2+3$. find the pre-images of each of the following under f.
(i) 19 (ii) 28 (iii) 2.

Solution:- given that $f(x) = x^2+3$

(i) $f(x) = 19$ then
$x^2+3 = 19$
$x^2 = 16$
$\boxed{x = \pm 4}$ Ans

(ii) $f(x) = 28$ then
$x^2+3 = 28$
$x^2 = 25$
$\boxed{x = \pm 5}$

(iii) $f(x) = 2$
$\therefore x^2+3 = 2$
$\boxed{x^2 = -1}$

So, 2 does not have any preimages under f.

Ans

Inverse of a function :-

Let $f : X \to Y$ and let $y \in Y$. Then we define,

$f^{-1}(y) = \{x \in X : f(x) = y\}$ = Set of preimages of y.

f^{-1} is called the inverse of f.

Question (10) :- Let $f : R \to R : f(x) = x^2 + 1$. Find (i) $f^{-1}(-4)$ (ii) $f^{-1}(10)$ (iii) $f^{-1}(5, 17)$.

Solution :- It is given that

$$f(x) = x^2 + 1$$

(i) let $f^{-1}(-4) = x$ Then

$$f(x) = -4$$
$$x^2 + 1 = -4$$
$$\boxed{x^2 = -5}$$

But, there is no real value of x

$\therefore f^{-1}(-4) = \phi$ Ans.

(ii) let $f^{-1}(10) = x$

$$\therefore f(x) = 10$$
$$x^2 + 1 = 10$$
$$x^2 = 9$$
$$\boxed{x = \pm 3}$$ Ans

(iii) Let $f^{-1}(5) = x$ then

$$f(x) = 5$$
$$x^2 + 1 = 5$$
$$x^2 = 4$$
$$\boxed{x = \pm 2}$$ Ans.

(iv) Let $f^{-1}(17) = x$

$$f(x) = 17$$
$$x^2 + 1 = 17$$
$$x^2 = 16$$
$$\boxed{x = \pm 4}$$ Ans