Matrices ⇒.

- Matrix is an Arrangement (It has No value).
- Matrices may have Real No. Complex No. or f^n.

Definition -

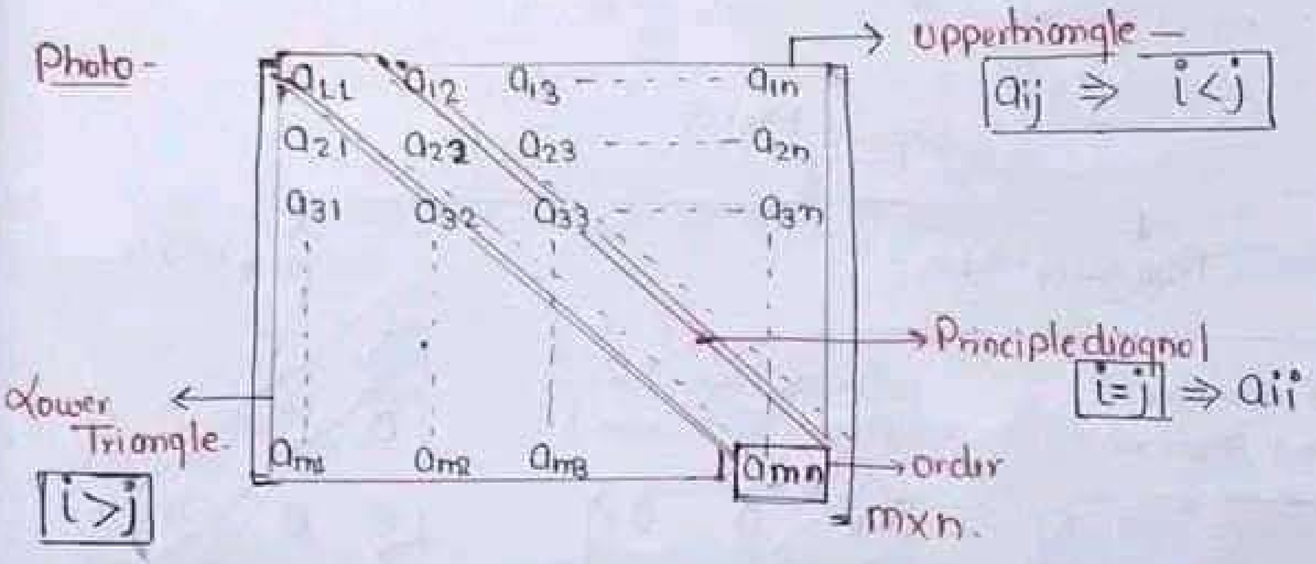
A Set of $m \cdot n$ Numbers (Real or complex) Arranged in Rectangular Array having $\boxed{m \text{ Rows}}$ & $\boxed{n \text{ Coulmn}}$ is called Matrix of $\boxed{\text{order } m \times n}$

$A = [a_{ij}]_{m \times n}$ → No. of Coulmn.

Element in matrix ← a_{ij} ; m → No. of Rows

a_{ij} : i^{th} Row, j^{th} Coulmn.

Photo -



Q find matrix of 3x2 order where $a_{ij} = \left|\frac{i-2j}{3}\right|$.

$$\begin{bmatrix} \frac{1}{3} & 1 \\ 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}_{3\times 2}$$

$a_{11} = \left|\frac{-1}{3}\right| = \frac{1}{3}$

$a_{12} = \left|\frac{1-4}{3}\right| = |-1| = 1$

* In $\boxed{m \times n \text{ order}}$ matrix No. of Element $= m \cdot n$.

Q if matrix have 12 Element then Possible orders will be -?

(12x1) (1x12) (6x2) (2x6) (3x4) (4x3)

↳ 6 different matrix possible

Types of Matrix (m×n)

- $m=1$ — Row Matrix
- $n=1$ — Coulmn Matrix
- $m>n$ — Vertical Matrix
- $m<n$ — Horizontal Matrix
- $m=n$ — Sq^r Matrix $n \times n$

{ m: No of Rows }
{ n: No. of Coulmn }

Square Matrix

- Triangular Matrix
  - Upper Triangular [upper Δ a^n Element $\bar{o}$]
  - Lower Triangular [lower Tri. a^n Element]
- diagonal Matrix.

Upper Triangular:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$\boxed{a_{ij}=0 \text{ if } i>j}$

Lower Triangular:

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\boxed{a_{ij}=0 \text{ if } i<j}$

diagonal Matrix:

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$\boxed{a_{ij}=0 \text{ if } i\neq j}$

Note

① No Element of principle diagonal in a diagonal matrix is Zero.

② Min No. of zeros in diag. Matrix is $= n(n-1)$
$= n^2-n$

eg $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \text{diag}(a, b, c)$

eg $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \text{diag}(-1, 2)$

diagonal Matrix –

Scalar Matrix

if diagonal matrix have Same/equal Element in diagonal

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \quad a_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ \underbrace{a}_{\text{Scalar}} & \text{if } i = j \end{cases}$$

$a \neq 1$

eg $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ or $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ or $[7]$

Unit matrix / Idenlity matrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

if $a=1$ $\quad a_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

⇒ dinoted As "I_n" ↳ if nxn order

eg

$I_1 = [1]_{1\times1}$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

⇒ **Singletom matrix**

if it has only one Element.

$A = [a_{ij}]_{m\times n}$ ⇒ Singleton if $\boxed{m=n=1}$

eg $[3]$, $[k]$

⇒ **Null matrix / Zero Matrix** if all Element are Zero

$A = [a_{ij}]$ ⇒ Null matrix if $\boxed{a_{ij} = 0}$

eg $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{2\times3}$ $\qquad O_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

⇒ **Sub Matrix**

matrix oblained by deleting any No. of Rows & No. of Coulmns.

eg $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$ is Sub matrix of $\begin{bmatrix} 0 & 9 & 5 \\ 2 & 3 & 4 \\ 8 & -2 & 5 \end{bmatrix}$

Trace of a Matrix ⇒ [only possible in Sq.r. matrix]

① Sum of all Element of diag. are Known As Trace of matrix.

② dinoted by $Tr(A)$.

③ $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad Tr(A) = a_{11}+a_{22}+a_{33}$

$$\boxed{Tr(A) = \sum_{i=1}^{n} a_{ii}}$$

Prop. of Tr(A)

let $A = [a_{ij}]_{n \times n}$, $B = [b_{ij}]_{n \times n}$

① $Tr(KA) = K \cdot Tr(A)$

② $Tr(A \pm B) = Tr(A) \pm Tr(B)$.

③ $Tr(AB) = Tr(BA)$

④ $Tr(A^n) = n\,Tr(A)$.

⑤ $Tr(I_n) = n$.

⑥ $Tr(AB) \neq Tr(A) \cdot Tr(B)$

⑦ $Tr(A) = Tr(CAC^{-1})$

$C \Rightarrow$ any NonSingular matrix

Q if $A+2B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 3 \\ 5 & 3 & 1 \end{bmatrix}$ & $2A-B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$.

Then find $Tr(A) - Tr(B) = ?$

$Tr(A+2B) = 1-3+1 = -1 \Rightarrow Tr(A) + 2Tr(B) = -1$ —— ①

$Tr(2A-B) = 3. \quad \Rightarrow 2Tr(A) - Tr(B) = 3 \quad \times 2$

Again $\quad Tr(A) + 2Tr(B) = -1$

$\quad 4Tr(A) - 2Tr(B) = 6$

$5Tr(A) = 5$

$\boxed{Tr(A) = 1}$

$\boxed{Tr(B) = -1}$

$\boxed{Tr(A) - Tr(B) = 2}$

Determinant of Matrix [Square Matrix].

if $A=[a_{ij}]_{n\times n} \Rightarrow$ then determinant of matrix $=|A|$

Note
① $|A_1 \cdot A_2 A_3 \cdots A_n| = |A_1||A_2||A_3|\cdots|A_n|$

② if K is a Scalar then $\boxed{|KA| = K^n|A|}$ *

⇒ Singular & Non Singular Matrix —

if $|A|=0 \Rightarrow$ Singular Matrix

$|A|\neq 0 \Rightarrow$ Non Singular Matrix –

Q find x if $\begin{bmatrix} 3 & -1+x & 2 \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{bmatrix}$ is Singular $x=?$ 0, −4

⇒ Equality of Matrix –

① They are of Same order

$A=(a_{ij})_{\boxed{m\times n}}$ | $B=(b_{ij})_{\boxed{p\times q}}$

$A=B$ if order Same $\boxed{\begin{matrix} m=p \\ n=q \end{matrix}}$ & $\boxed{a_{ij}=b_{ij}}$

② element in Corresponding position are equal —

eg if $\begin{bmatrix} x-4 & 2y+2 \\ 2x-4 & 2z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$ then $x, y, z, w = ?$

$x=1$, $y=2$, $z=3$, $w=7$.

⇒ Scalar Multiplication of matrix –

① ↳ if a Scalar/Konstant is multiplied then it is Multiplied to each Element of Matrix —

$$\boxed{k(a_{ij})_{m\times n} = (k\, a_{ij})_{m\times n}}$$

eg $2\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$

② $2 \cdot \text{diag}(a_1, a_2, a_3) = \text{diag}(2a_1, 2a_2, 2a_3)$.

⇒. Multiplication of Matrices

① if $A = (a_{ij})_{m\times n}$ & $B = (b_{ij})_{n\times p}$

match Then only multiplication possible

② if $A = (a_{ij})_{m\times n}$ & $B = (b_{ij})_{p\times q}$. & AB is possible — then —

$\boxed{n = p}$

Q if $\underset{1\times3}{[x \cdot y\ z]} \underset{3\times3}{\begin{bmatrix} a & b & c \\ p & q & r \\ s & t & v \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}$ possible then = ?

Prod. $= [\quad]_{1\times1}$

Q if $A = \text{diag}(1,2,3)$ & $B = \text{diag}(5,6,7)$ check $AB = BA$

$AB = \text{diag}(5, 12, 21)$

$BA = \text{diag}(5, 12, 21)$

$\boxed{AB = BA}$

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③ In Normal matrix

Generally $\boxed{AB \neq BA}$

Q $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 1 \end{bmatrix}_{3\times2}$ & $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}_{3\times3}$ which one is defined in AB & BA

$\begin{cases} AB \Rightarrow \text{Not defined} \\ BA \Rightarrow \text{defined} \end{cases}$

Q if M is matrix of 3x3 order such that —

$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ & $M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ & $M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$ find M = ?

Let $M = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$

$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{matrix} a_2 = -1 \\ a_5 = 2 \\ a_8 = 3 \end{matrix}$

$M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} a_1 - a_2 = 1 \\ a_4 - a_5 = 1 \\ a_7 - a_8 = -1 \end{cases}$

$M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} \Rightarrow \begin{cases} a_1 + a_2 + a_3 = 0 \\ a_4 + a_5 + a_6 = 0 \\ a_7 + a_8 + a_9 = 12 \end{cases}$

Q $A=\begin{bmatrix}\omega & 0\\ 0 & \omega\end{bmatrix}$ then $A^{70}=?$

$$A^2=\begin{bmatrix}\omega & 0\\ 0 & \omega\end{bmatrix}\times\begin{bmatrix}\omega & 0\\ 0 & \omega\end{bmatrix}=\begin{bmatrix}\omega^2 & 0\\ 0 & \omega^2\end{bmatrix}$$

$$A^{70}=\begin{bmatrix}\omega^{70} & 0\\ 0 & \omega^{70}\end{bmatrix}=\begin{bmatrix}\omega & 0\\ 0 & \omega\end{bmatrix}$$

* also if $\boxed{A=\text{diag}\cdot(a\cdot b\ c)}$

Then $\boxed{A^n=\text{diag}\cdot(a^n\ b^n\ c^n)}$

Prop. of Multiplication -

① if $AB=0 \not\Rightarrow A=0$ or $B=0$

But $AB=0 \Rightarrow$ either $|A|=0$ or $|B|=0$ or $A\&B\neq 0$

But Not Conversetrue -

② $AB\neq BA$ (Generally)

$AB=BA$ (Commute)

$AB=-BA$ (anti commute)

③ $A(BC)=(AB)C$

④ $A\cdot(B+C)=AB+AC$

⑤* $\boxed{AI=IA=A} \Rightarrow \boxed{I^2=I^3=\dots=I^n}$

⑥ $AB=AC \not\Rightarrow \boxed{B=C}$

⑦ $(A+B)^2=A^2+B^2+AB+BA$

$(A-B)^2=A^2+B^2-AB-BA$

$(A-B)(A+B)=A^2+B^2+AB-BA$

$(A+B)(A-B)=A^2-AB+BA-B^2$

$A(-B)=(-A)B=-AB$

⑧ $I^n=I$

Some Special Multiplication -

$A^2 = I \Rightarrow$ Involutory Matrix

$A^2 = A \Rightarrow$ Idimpotent matrix

$A^k = 0 \Rightarrow$ Nilpotent matrix

$A^{k+1} = A \Rightarrow$ periodic matrix

* $\boxed{A^T \cdot A = I} \Rightarrow$ orthogonal matrix

Q $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is Involutory or Not?

Q $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is Nilpotent ?

Q if $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

Then $tr(A) + tr\left(\frac{ABC}{2}\right) + tr\left(\frac{A(BC)^2}{4}\right) + \cdots \infty = ?$

$BC = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\boxed{BC = I}$ $\quad A \cdot BC = A \cdot I = A$

$A(BC)^2 = A \cdot I^2 = A$

So $\quad tr(A) + tr\left(\frac{A}{2}\right) + tr\left(\frac{A}{4}\right) + \cdots \infty$

$= tr(A)\left[1 + \frac{1}{2} + \frac{1}{4} + \cdots\right]$

$= tr(A) \cdot \frac{1}{1 - 1/2} = 2tr(A)$ $\qquad tr(A) = 2 + 1 = 3$

$= \underline{6}$

Transpose of matrix

① Represented by A', A^T, or A^t.

② if $A = (a_{ij})_{m\times n}$ then $A^T = (a_{ji})_{n\times m}$

Transpose means changing Row $\rightleftharpoons$ Coulmn.

eg-

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 5 & 6 & 7 & 4 \\ 0 & 8 & 9 & 3 \end{bmatrix}_{3\times 4} \qquad A^t = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 6 & 8 \\ 3 & 7 & 9 \\ 2 & 4 & 3 \end{bmatrix}_{4\times 3}.$$

Prop of $A^T \Rightarrow$

① $(A^T)^T = A$

② $(A \pm B)^T = A^T \pm B^T$

③ $(A \cdot B)^T = B^T A^T$ (Reversal Law of Transpose)

④ $(A_1 A_2 A_3 \cdots A_n)^T = (A_n^T A_{n-1}^T \cdots A_3^T A_2^T A_1^T)$

⑤ $(KA)^T = K(A)^T$.

⑥ $(A^n)^T = (A^T)^n$.

* if $A = \text{diag}(a_1, a_2, a_3)$

Then $A^T = A = \text{diag}(a_1, a_2, a_3)$

* $\boxed{A^T}$ (Square matrix) $A = [a_{ij}]_{n\times n}$.

- Symmetric Matrix: $A^T = A$ → $\boxed{a_{ji} = a_{ij}}$
- Orthogonal matrix: $AA^T = I$
- Skew Sym. Matrix: $A^T = -A$

eg $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$,

* max No. of distinct Entries in Symm. matrix of order $n = \dfrac{n(n+1)}{2}$.

$(a_{ji}) = -(a_{ij}) \Rightarrow$ For principal diagonal Put $j = i$

$a_{ii} = -a_{ii}$

$2a_{ii} = 0 \Rightarrow$ * $\boxed{a_{ii} = 0}$

* In Skew Symm. Matrix all Element of principal diag. are always Zero.

eg $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

Q If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, $abc = 1$, $A^T A = I$ then find value of $a^3+b^3+c^3 = ?$

$|A|^2 = 1 \Rightarrow |A| = \pm 1$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \pm 1$$

$3abc - (a^3+b^3+c^3) = \pm 1$

$3 - (a^3+b^3+c^3) = \pm 1 \Rightarrow a^3+b^3+c^3 = 3 \pm 1$

$= \underline{\underline{2 \text{ or } 4}}$

$\Rightarrow$ Complex Conjugate of A matrix –

$\Rightarrow$ matrix having complex No. as its Element.

$\Rightarrow$ Then each Element Replace $A = a+ib$ by $\bar{A} = a-ib$ we get complex conjugate of matrix.

eg. $A = \begin{bmatrix} 2+5i & 3-i & 7 \\ -2i & 6+i & 7-5i \\ 1-i & 3 & 6i \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} 2-5i & 3+i & 7 \\ 2i & 6-i & 7+5i \\ 1+i & 3 & -6i \end{bmatrix}$

Prop.

① $\overline{(\bar{A})} = A$

② $\overline{(A+B)} = \bar{A} + \bar{B}$

③ $\overline{(KA)} = K\bar{A}$

④ $\overline{AB} = \bar{A}\bar{B}$

$\Rightarrow$ Conjugate transpose of matrix

① denoted by A^θ

② $A^\theta = \overline{(A^T)}$ = conjugate of A^T

eg. $A = \begin{bmatrix} 2+4i & 3 & 5-9i \\ 4 & 5+2i & 8i \\ 2 & -5 & 4-i \end{bmatrix} \Rightarrow A^\theta = \overline{(A^T)} = \begin{bmatrix} 2-4i & 4 & 2 \\ 3 & 5-2i & -5 \\ 5+9i & -8i & 4+i \end{bmatrix}$

Prop

① $\overline{(A^T)} = (\bar{A})^T$

② $(A^\theta)^\theta = A$

③ $(A+B)^\theta = A^\theta + B^\theta$

④ $(KA)^\theta = KA^\theta$

⑤ $(AB)^\theta = B^\theta A^\theta$.

A^{θ}

Hermition Matrix	Unitary matrix	Skew Hermition matrix
$A^{\theta} = A$	$AA^{\theta} = I$	$A^{\theta} = -A$

Hermition Matrix

① $a_{ij} = \overline{a_{ji}}$

Put $j=i$: $a_{ii} = \overline{a_{ii}}$

$\Rightarrow a_{ii}$ is purely Real.

all diagonal Element of Herm. must be Purely Real.

② $\Rightarrow (A+A^{\theta})^{\theta} = (A+A^{\theta})$

Unitary matrix

① $A^{-1} = A^{\theta}$

② if A, B unitary Then AB also unitary

③ $A \Rightarrow$ unitary $\Rightarrow A^{-1}$ & $\bar{A}$ also unitary

Skew Hermition matrix

① $a_{ij} = -\overline{a_{ji}}$

$j=i$ Put

$a_{ii} = -\overline{a_{ii}}$

$a_{ii} + \overline{a_{ii}} = 0$

$\Rightarrow a_{ii}$ is purely Img.

$\Rightarrow$ means diag. Element must be purly Img. or Zero.

② $(A-A^{\theta})^{\theta} = -(A-A^{\theta})$.

So $\Rightarrow$ if A is squar matrix

Then

$$\boxed{A = \underbrace{\frac{1}{2}(A+A^{\theta})}_{\text{Hermiti}} + \underbrace{\frac{1}{2}(A-A^{\theta})}_{\text{Skew Hermi}}}$$

Q Express A as Sum of Herm. & Skew Hermi.

$$A = \begin{bmatrix} 2+3i & 7 \\ 1-i & 2i \end{bmatrix}$$

$A = P + Q$ $\quad P = \frac{1}{2}(A+A^{\theta})$

$Q = \frac{1}{2}(A-A^{\theta})$.

Q. if A is SkewSymm then $A^n = ?$

$\hookrightarrow A^T = -A$

$B = A^n$

$B^T = (A^T)^n$

$= (-A)^n = \begin{cases} A^n \Rightarrow n = \text{even} \rightarrow \text{Symm} \\ -A^n \Rightarrow n = \text{odd} \rightarrow \text{Skew Symm} \end{cases}$

Q. if $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ then $A = P + Q$ where P is Sym, Q = SkewSym find P & Q

$P = \frac{1}{2}(A + A^T) = \frac{1}{2}\left[\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}\right]$

$Q = \frac{1}{2}(A - A^T)$

$\Rightarrow$ Determinant of Matrix $\Rightarrow$

① Represented by $\det(A)$ or $|A|$

② $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 5 - 6 = -1$

Prop of Det

*① $|A^T| = |A| \rightarrow$ exist $\Leftrightarrow$ if A is SquareMatrix.

② $|AB| = |BA|$

③ $|AB| = |A||B|$

**④ $\boxed{|KA| = K^n|A|}$ $\boxed{n = \text{order}}$ of Sq. Matrix

*⑤ if $A = \text{diag}(a_1\, a_2\, a_3)$

then $|A| = a_1 a_2 a_3$.

if $A = \text{diag}(a_1, a_2, a_3, \cdots, a_n)$ then $|A| = a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdots a_n$.

⑥ if A is orthogonal matrix $\Rightarrow$

$A^T \cdot A = I \Rightarrow |AA^T| = |I| \Rightarrow |A^T||A| = 1$

$\Rightarrow |A|^2 = 1 \Rightarrow \boxed{|A| = \pm 1}$

⑦ if A is SkewSymmMatrix $\Rightarrow$

$\boxed{\text{odd order}}$ $A^T = -A \Rightarrow |A| = -|A^T| \Rightarrow 2|A| = 0 \Rightarrow \boxed{|A| = 0}$

⑧ SkewSymm. of Even order Then $|A|$ = PerfectSquare.

Q if A is SkewSymm. then $A^n = ?$

↳ $A^T = -A$

$B = A^n$

$B^T = (A^T)^n$

$= (-A)^n = $ → $A^n \Rightarrow n = \text{even} \rightarrow$ Symm.

→ $-A^n \Rightarrow n = \text{odd} \rightarrow$ Skew Symm

Q if $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ then $A = P + Q$ where P is Sym, Q = Skew Sym find P & Q

$P = \frac{1}{2}(A + A^T) = \frac{1}{2}\left[\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}\right]$

$Q = \frac{1}{2}(A - A^T)$

⇒ Diterminant of Matrix ⇒

① Represented by $\det(A)$ or $|A|$

② $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 5 - 6 = -1$

Prop of Det.

*① $|A^T| = |A| \rightarrow$ exist $\Leftrightarrow$ if A is Square Matrix.

② $|AB| = |BA|$

③ $|AB| = |A||B|$

**④ $\boxed{|kA| = k^n|A|}$ $\boxed{n = \text{order}}$ of Sq.r Matrix-

*⑤ if $A = \text{diag.}(a_1\ a_2, a_3)$

then $|A| = a_1 a_2 a_3$.

if $A = \text{diag}(a_1, a_2, a_3, \dots, a_n)$ then $|A| = a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdots a_n$.

⑥ if A is orthogonal matrix ⇒

$A^T \cdot A = I \Rightarrow |AA^T| = |I| \Rightarrow |A^T||A| = 1$

$\Rightarrow |A|^2 = 1 \Rightarrow \boxed{|A| = \pm 1}$

⑦ if A is SkewSymm Matrix ⇒

$\boxed{\text{odd order}}$ $A^T = -A \Rightarrow |A| = -|A^T| \Rightarrow 2|A| = 0 \Rightarrow \boxed{|A| = 0}$

⑧ SkewSymm. of Even order Then $|A|$ = Perfect Square.

⑨ Singular Matrix $\Rightarrow |A| = 0 \Rightarrow$ Inverse doesnot Exist

⑩ Non Singular Matrix $\Rightarrow |A| \neq 0 \Rightarrow$ Invertible or A^{-1} exist

⑪ interchanging two Rows/Coulmn changes Sign of $|A|$

⑫ if all Elements of any Row/Coulmn are Zero Then $|A| = 0$

⑬ if any two Rows/Coulmn are Same then $|A| = 0$

⑭ $|A|^n = |A^n|$

$|A^{-1}| = |A|^{-1} = \frac{1}{|A|}$

⑮ if one Row or Coulmn is multiplied by "K" then $|A|$ is multiplied by "K" —

$\underbrace{|KA| = K^n|A|}_{\text{only one Row/coulmn}}$ But $\underbrace{[KA] = K[A]}_{\text{Every Element}}$

⑯ determinant of Triangular Matrix equal to product of diagonal Elements.

⑰ Orthogonal Matrix $\Rightarrow AA^T = I$

$\Rightarrow |AA^T| = 1 \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

⑱ involutary Matrix $\Rightarrow |A| = \pm 1$

Idempotent " $\Rightarrow |A| = 0, 1$

Skew Symm " $\Rightarrow |A| = 0$ or perfect Square

⑲ $|A| = \sum_{k=1}^{n} a_{ik} A_{ik} = \sum_{k=1}^{n} a_{kj} A_{kj}$

A_{ik}: Cofactor of A, Same A_{kj}: Cofactor of A, Same

$0 = \sum_{k=1}^{n} a_{ik} A_{jk} = \sum_{k=1}^{n} a_{ki} A_{kj}$

different   different

ADjoint of Matrix -

① Rep. by Adj(A).

& let C_{ij} be Cofactor of a_{ij} in A

Then transpose of matrix of Cofactors of Element A is Called Adjoint of A.

② $\text{Adj}(A) = [C_{ij}]^T$

$\Rightarrow (\text{adj}A)_{ij} = C_{ji}$ $\Rightarrow$ eg. $(\text{adj}A)_{32} = (C_{ij})_{23}$.

eg if $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & -1 & 2 \end{bmatrix} \Rightarrow \text{Adj}A = \begin{bmatrix} 7 & -15 & -10 \\ 3 & -3 & -6 \\ 1 & 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 7 & 3 & 1 \\ -15 & -3 & 3 \\ -10 & -6 & 2 \end{bmatrix}$.

Prop. of Adjoint A -

①** $|\text{adj}\,A| = |A|^{n-1}$. for NonSingular $|A| \neq 0$

Proof

② $A(\text{adj}\,A) = (\text{adj}\,A)\,A = |A|I_n$.

③ $A(\text{adj}A) = (\text{adj}A)A = 0$ if $|A| = 0 \Rightarrow$ Singular matrix

$\boxed{\text{Proof 1}}$

$A(\text{adj}A) = |A|\,I_n$

taking det. Both Sides ——

$|A(\text{adj}\,A)| = ||A|I_n|$

$|A||\text{adj}\,A| = |A|^n |I_n| = |A|^n$.

$\boxed{|\text{adj}\,A| = |A|^{n-1}}$ **

④ $|\text{adj}(\text{adj}(\text{adj} \cdots \text{adj}(A)))| = |A|^{(n-1)^m}$

$\longleftarrow$ adj Repeat m times $\longrightarrow$

eg $|\text{Adj}(\text{adj}A)| = |A|^{(n-1)^2}$.

⑤ $(\text{adj}A)^T = \text{adj}A^T$

⑥ $(\text{Adj}\,A^n) = (\text{Adj}\,A)^n$

⑦ $\text{Adj}(KA) = K^{n-1}(\text{adj}A)$

<u>Proof</u> $\quad A(\text{adj}A) = |A|I_n$

Replace $A \to KA$

$$KA(\text{adj}(KA)) = |KA|I_n$$
$$= K^n|A|I_n$$
$$A(\text{adj}(KA)) = K^{n-1}\underline{|A|I_n}$$
$$= K^{n-1}A(\text{adj}A).$$
$$\boxed{\text{adj}(KA) = K^{n-1}(\text{adj}A)}$$

⑧ $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$

⑨ $\text{adj}(\text{adj}A) = |A|^{n-2}\cdot A.$

<u>Proof</u> $\quad A\cdot(\text{adj}A) = |A|I_n$

Replace $A \to \text{adj}A \Rightarrow (\text{adj}A)(\text{adj}(\text{adj}A)) = |\text{adj}A|I_n$

$$= |A|^{n-1}I_n.$$

multiply Both Side by A

$$\Rightarrow \underbrace{A(\text{adj}A)}(\text{adj}(\text{adj}A)) = AI_n|A|^{n-1}$$
$$|A|I_n(\text{adj}(\text{adj}A)) = A\cdot|A|^{n-1}.$$
$$\boxed{(\text{adj}(\text{adj}A)) = |A|^{n-2}A}$$

also $\quad |\text{adj}(\text{adj}A)| = |A|^{(n-1)^2}.$

⑩ $|A\,\text{Adj}A| = |A|^n.$

⑪ <u><u>Q</u></u> if $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}_{n=3}$ then $|A||\text{adj}A| = ?$

$$|A||\text{adj}A| = |A||A|^2 = |A|^3$$
$$= (a^3)^3 = \underline{\underline{a^9}}$$