Chapter 4

DETERMINANTS

POINTS TO REMEMBER:

Determinant:

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, where a_{ij} = i^{th} row, j^{th} column element of A.

Definition of determinants in terms of function:

If M is the set of square matrices, K is the set of numbers (real or complex) and f : M → K is defined by f (A) = k, where A ∈ M and k ∈ K, then f(A) is called the determinant of A. It is also denoted by | A| or det A or Δ.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then determinant of A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$

Remarks:

(i) For matrix A, | A| is read as determinant of A and not modulus of A.

(ii) Only square matrices have determinants.

Properties of Determinants:

Property 1: The value of the determinant remains unchanged if its rows and columns are interchanged.

Property 2: If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

Property 3: If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.

Property 4: If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

Property 5: If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

Property 6: If each element of a row or column is added with the multiple of another row or element, then the determinant of the matrix remains unchanged.

Key points to remember about determinants:

(1) Let $A = [a_{ij}]$ of order n, then $|kA| = k^n\ |A|$

(2) If A and B are square matrix of the same order, then $|AB| = |A|\ |B|$.

(3) Let $A = [a_{ij}]$ is a diagonal matrix (lower triangular matrix or upper triangular matrix or scalar matrix) of order n $(n \geq 2)$, then

$|A| = a_{11} \times a_{22} \times a_{33} \times \dots \times a_{nn}$

Area of a Triangle:

Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by the expression $\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$ square units.

Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Remarks:

(i) Since area is a positive quantity, we always take the absolute value of the determinant.

(ii) If area is given, use both positive and negative values of the determinant for calculation.

(iii) The area of the triangle formed by three collinear points is zero and vice versa.

Minors and Co-factors

Definition: Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i[th] row and j[th] column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}

Remark: Minor of an element of a determinant of order n ($n \geq 2$) is a determinant of order $n - 1$.

Definition: Co-factor of an element a_{ij}, denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is minor of a_{ij}.

Note: If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

For example, $\Delta = a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = 0$.

Adjoint of a matrix:

The adjoint of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n\times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by adj A.

Let $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $\text{adj}A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

Theorem 1:

If A be any given square matrix of order n, then $A(\text{adj } A) = (\text{adj } A)A = |A|I$, where I is the identity matrix of order n.

Definition: A square matrix A is said to be singular, if $|A| = 0$.

A square matrix A is said to be non-singular, if $|A| \neq 0$

Theorem 2: If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

Theorem 3: The determinant of the product of matrices is equal to product of their respective determinants, i.e. $|AB| = |A|\,|B|$, where A and B are square matrices of the same order.

Theorem 4: A square matrix A is invertible if and only if A is a non-singular matrix.

$A^{-1} = \frac{adj\ A}{|A|}$ where $|A| \neq 0$.

Some important points to remember related to adjoint and inverse of a Matrix.

(1) Let A be a square matrix of order n, then $|adj\ A| = |A|^{n-1}$.

(2) If A and B are square matrices of same order, then
adj (AB) = (adj B) × (adj A).

(3) If A is an invertible square matrix, then $adj\,(A^T) = (adj\ A)^T$.

(4) Let A be a square matrix of order n, then $adj(adj\ A) = |A|^{n-2}A$.

(5) Let A be a non-singular square matrix, then $|A^{-1}| = \frac{1}{|A|}$.

(6) If A and B are non-singular square matrices of same order, then
$(AB)^{-1} = (B)^{-1}(A)^{-1}$

(7) If A is an invertible square matrix, then A^T is also invertible and
$(A^T)^{-1} = (A^{-1})^T$

(8) If A is an invertible square matrix, then $A A^{-1} = A^{-1}A = I$ and $(A^{-1})^{-1} = A$.

Solution of system of linear equations using inverse of a matrix:

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then, the system of equations can be written as, AX = B, i.e.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case I: If A is a non-singular matrix, then its inverse exists.

$X = A^{-1} B$

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as Matrix Method.

Case II: If A is a singular matrix, then $|A| = 0$. In this case, we calculate (adj A) B.

If (adj A) B ≠ O, (O being zero matrix), then solution does not exist and the system of equations is called inconsistent.

If (adj A) B = O, then system may be either consistent or inconsistent accordingly the system have either infinitely many solutions or no solution.