CHAPTER 6

APPLICATION OF DERIVATIVES

Points to remember:

Rate of change: Whenever one quantity y varies with respect to another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $f'(x_1)$ or $\frac{dy}{dx}$ at $x = x_1$, represents the rate of change of y with respect to x at $x = x_1$.

Increasing and decreasing functions:

Let I be an interval in the domain of a real valued function f. Then f is said to be

(i) Increasing on I, if $x_1 < x_2$ in I $\Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in$ I.

(ii) Strictly increasing on I, if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

(iii) Decreasing on I, if $x_1 < x_2$ in I $\Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in$ I.

(iv) Strictly decreasing on I, if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in$ I.

(v) Constant on I, if $f(x) = c$ for all $x \in$ I, where c is a constant.

Derivative test:

Let f be continuous on [a, b] and differentiable on (a,b), then

(a) f is increasing in [a,b], if $f'(x) > 0$ for each $x \in (a, b)$

(b) f is decreasing in [a,b], if $f'(x) < 0$ for each $x \in (a, b)$

(c) f is a constant function in [a,b] if $f'(x) = 0$ for each $x \in (a, b)$

Maxima and Minima:

Let f be a function defined on an interval I. Then

(a) f is said to have a maximum value in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in$ I. The value f(c) is called the maximum value of f in I and the point c is called a point of maximum value of f in I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that f (c) < f (x), for all $x \in$ I. The value f (c), in this case, is called the minimum value of f in I and the point c, in this case, is called a point of minimum value of f in I.

(c) f is said to have an extreme value in I if there exists a point c in I such that f (c) is either a maximum value or a minimum value of f in I. The value f (c), in this case, is called an extreme value of f in I and the point c is called an extreme point.

By a monotonic function f in an interval I, we mean that f is either increasing in I or decreasing in I.

Every continuous function on a closed interval has a maximum and a minimum value.

Theorem: Let f be a function defined on an open interval I. If f has a local maxima or a local minima at x = c, then either f ′(c) = 0 or f is not differentiable at c , where c ∈ I.

Critical points: A point c in the domain of a function f at which either f ′(c) = 0 or f is not differentiable is called a critical point of f.

Theorem (First Derivative Test): Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I, then

(i) If f ′(x) changes sign from positive to negative as x increases through c, i.e. if f ′(x) > 0 at every point sufficiently close to and to the left of c, and f ′(x) < 0 at every point sufficiently close to and to the right of c, then c is a point of local maxima.

(ii) If f ′(x) changes sign from negative to positive as x increases through c, i.e. if f ′(x) < 0 at every point sufficiently close to and to the left of c, and f ′(x) > 0 at every point sufficiently close to and to the right of c, then c is a point of local minima.

(iii) If f ′(x) does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. In fact, such a point is called point of inflexion.

If c is a point of local maxima of f, then f(c) is a local maximum value of f.

Similarly, if c is a point of local minima of f, then f(c) is a local minimum value of f.

Theorem (Second Derivative Test): Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c, then

(i) $x = c$ is a point of local maxima, if $f'(c) = 0$ and $f''(c) < 0$.
The value f (c) is local maximum value of f.

(ii) $x = c$ is a point of local minima, if $f'(c) = 0$ and $f''(c) > 0$.
In this case, f (c) is local minimum value of f.

(iii) The test fails, if $f'(c) = 0$ and $f''(c) = 0$.
In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

Theorem: Let f be a continuous function on an interval $I = [a, b]$. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

Theorem: Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

(i) $f'(c) = 0$, if f attains its absolute maximum value at c.

(ii) $f'(c) = 0$, if f attains its absolute minimum value at c.