Mathematical tools

1. Equations in Physics

☑ What is an equation?

An equation states that two expressions are equal. In physics, equations describe how quantities relate to each other.

◇ Types of Equations in Physics

Type	Example	Meaning
Kinematic equations	v=u + at	Final velocity in linear motion
Dynamic equations	F=ma	Newton's Second Law
Energy equations	$E=1/2mv^2$	Kinetic energy
Wave equations	v=fλv	Speed of a wave
Electric equations	V=IR	Ohm's Law

◇ How to Use Equations in Physics

1. **Substitute known values** (e.g. $a=9.8\ m/s^2$)
2. **Solve for unknown** using algebra
3. **Check units** to ensure correctness
4. **Analyse the result** physically

◇ 2. Inequalities in Physics

☑ What is an inequality?

An inequality shows that one quantity is **greater than**, **less than**, or **not equal to** another.

◇ Examples in Physics

Inequality	Meaning
v<c	Nothing can move faster than light (special relativity)
$\mu_s \geq \tan\theta$	Condition for static friction preventing slipping
KE≤EKE	Kinetic energy ≤ total energy in bound systems
P>0	Pressure is always positive in thermodynamic systems
E>WE	Energy input must exceed useful work in real machines (due to losses)

◇ How to Use Inequalities

- **Set boundaries or conditions** (e.g., for equilibrium or stability)
- **Predict feasibility** (e.g., whether motion will occur)
- **Estimate maximum or minimum values** (e.g., max height, max speed)

▧ Mathematical Tool: Quadratic Equations in Physics

A **quadratic equation** is an equation of the form:

$ax^2+bx+c=0$

where:

- a, b, c are constants
- x is the variable
- $a\neq0$

- ## 🔹 1. General Solution (Quadratic Formula)

- To solve $ax^2+bx+c=0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The term under the square root, b^2-4ac is called the **discriminant**:

- If **positive**, two real roots
- If **zero**, one real root

- If **negative**, no real roots (complex roots)
- # 2. Quadratic Equations in Physics
- ✅ **Common Physics Contexts**

Context	Example	Physical Meaning
Kinematics	$s = ut + \frac{1}{2}at^2$	Solving for time t
Projectile motion	$y = y_0 + v_0t - \frac{1}{2}gt^2$	Solving for height or time
Conservation of energy	$\frac{1}{2}mv^2 + mgh = E$	Solving for speed v
Lenses/Mirrors	$\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$	Can be rearranged into quadratic form

4. Quick Tips

- Always check units.
- Use the **discriminant** to know the nature of the solution.
- In physics, **ignore negative time or non-physical roots**.

📘 Mathematical Tool: Proportionality and Scaling in Physics

- **Proportionality** and **scaling** are powerful tools in physics to understand how one quantity changes when another changes.

1. Types of Proportionality

✅ Direct Proportionality

- $A \propto B$
- $\Rightarrow A = kB$ \
- As B increases, A increases.
- Graph: Straight line through the origin.
- **Examples:**
- $F \propto a$ (Newton's second law)
- $V \propto T$ (at constant pressure, ideal gas law)

✅ Inverse Proportionality

$$A \propto \frac{1}{B} \quad \Rightarrow \quad A = \frac{k}{B}$$

As B increases, A decreases.

Examples:

- $F \propto \frac{1}{r^2}$ (gravitational/electric force)
- $P \propto \frac{1}{V}$ (Boyle's Law)

✅ Quadratic or Power Proportionality

$$A \propto B^n$$

Examples:

- $E_k \propto v^2$ (kinetic energy)
- $I \propto r^2$ (intensity of light from a point source)
- $T \propto L^{1/2}$ (simple pendulum)

2. Scaling in Physics

What is Scaling?

Scaling shows how one physical quantity changes when dimensions or input values change.

🔸 Scaling Laws

Quantity	Scales as	Example
Area	L^2	Doubling length → area becomes 4 times
Volume	L^3	Tripling length → volume becomes 27 times
Mass	L^3 (assuming same density)	Larger objects have much more mass
Gravitational Force	$\propto \frac{1}{r^2}$	Force decreases quickly with distance
Drag Force	$\propto v^2$	Faster object faces more resistance

▧ Mathematical Tool: Logarithmic and Exponential Relationships in Physics

Exponential and logarithmic functions are used in physics to describe rapid changes—like **growth**, **decay**, and **scaling** over time or space.

◇ 1. Exponential Relationships

☑ General Form:

$y = Ae^{kx}$

Where:

- A: initial value
- e: Euler's number (≈2.718)
- k constant (positive → growth, negative → decay)
- x variable (time, distance, etc.)

🔸 Examples in Physics

Situation	Equation	Description
Radioactive decay	$N = N_0 e^{-\lambda t}$	N: remaining nuclei, λ: decay constant
Charging a capacitor	$Q = Q_0(1 - e^{-t/RC})$	Exponential approach to max charge
Cooling (Newton's Law)	$T = T_0 e^{-kt}$	Temperature decays over time
Population growth	$P = P_0 e^{rt}$	In ecology or thermodynamics
Sound intensity loss	$I = I_0 e^{-\alpha x}$	Damping in a medium

2. Logarithmic Relationships

✅ **General Form:**

x=log $_b$(y)

Where:

- log $_b$: logarithm base b
- ln(y) : natural logarithm (base e)

🔸 Examples in Physics

Situation	Equation	Description
Sound level (decibels)	$L = 10 \log_{10}\left(\frac{I}{I_0}\right)$	Logarithmic scale for sound intensity
pH scale	$\text{pH} = -\log_{10}[H^+]$	In chemistry/biology
Richter scale (earthquakes)	$M = \log_{10}\left(\frac{A}{A_0}\right)$	Measures magnitude on a logarithmic scale
Half-life calculations	$t = \frac{\ln(N_0/N)}{\lambda}$	Logarithmic decay

🔹 **3. Key Properties**

🔸 **Exponential Rules**

🔸 Exponential Rules

- $e^a \cdot e^b = e^{a+b}$
- $(e^a)^b = e^{ab}$
- $e^0 = 1$

🔸 Logarithmic Rules

- $\log(ab) = \log a + \log b$
- $\log(a^b) = b \log a$
- $\log\left(\frac{a}{b}\right) = \log a - \log b$

🔹 4. Visual Understanding

- **Exponential growth**: rapid rise (J-shaped curve)
- **Exponential decay**: rapid fall leveling off
- **Logarithmic**: fast increase at first, then slows down (opposite of exponential)

2. Trigonometry

📘 1. Sine, Cosine, Tangent — Basic Trigonometric Ratios

These relate the angles of a right triangle to the ratios of its sides:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

✅ Applications in Physics:

Concept	Use
Inclined planes	Resolve forces: $F_{\|} = mg\sin\theta$, $F_{\perp} = mg\cos\theta$
Projectiles	Resolve velocity: $v_x = v\cos\theta$, $v_y = v\sin\theta$
Circular motion	Components of radial and tangential acceleration

📘 2. Trigonometric Identities

These simplify and solve equations involving angles.

✅ Fundamental Identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

✅ Common Uses in Physics:

- **Simplifying wave equations** (e.g., A cos (ωt), A cos (ωt) and
- **Phase differences** in interference
- **Solving motion on inclined surfaces** or pendulum motion

📘 3. Vector Components and Angles

- ✅ **Resolving a Vector into Components:**
- For a vector $\vec{A}$ at angle θ to the horizontal

$$A_x = A\cos\theta, \quad A_y = A\sin\theta$$

- A_x: horizontal component
- A_y: vertical component

✅ Rebuilding from Components:

$$A = \sqrt{A_x^2 + A_y^2}, \quad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

✅ **Used in:**

- **Projectile motion**
- **Forces and torque**
- **Electric fields**
- **Work done by angled forces** W=F d cos θ

📘 4. Circular Motion and Waves

- ✅ **Circular Motion**

✅ Trigonometry in Circular Motion

- Position as function of time:

$$x(t) = r\cos(\omega t), \quad y(t) = r\sin(\omega t)$$

- Useful in describing **uniform circular motion** and **simple harmonic motion** (SHM)

Angle Sum and Difference Formulas

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Double Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Half-Angle Formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Product-to-Sum Formulas

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

Sum-to-Product Formulas

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Values of Trig Functions at Standard Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	Not defined

❖ Vectors in Physics

Vectors are quantities that have both **magnitude** and **direction**. They're essential in physics to represent quantities like **force**, **velocity**, **acceleration**, and **displacement**

📘 1. Vector Addition and Subtraction

✅ Geometrical (Graphical) Method:

- **Head-to-tail method** for addition
- Tip-to-tip subtraction: $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

✅ Algebraic Method:

If

$$\vec{A} = A_x\hat{i} + A_y\hat{j}, \quad \vec{B} = B_x\hat{i} + B_y\hat{j}$$

Then:

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

$$\vec{A} - \vec{B} = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j}$$

📘 2. Dot Product (Scalar Product)

✅ Definition:

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

Where θ is the angle between $\vec{A}$ and $\vec{B}$.

✅ Component Form:

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

✅ Properties:

- Scalar result
- Commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- Used to find angle between vectors:

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{AB}$$

✅ Applications:

- **Work done**: $W = \vec{F} \cdot \vec{d} = Fd\cos\theta$
- **Projection** of one vector onto another

📘 3. Cross Product (Vector Product)

✅ **Definition:**

$$\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$$

Where:

- $\hat{n}$ is a unit vector **perpendicular** to both $\vec{A}$ and $\vec{B}$
- Follows the **right-hand rule**

✅ **Component Form:**

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

✅ **Properties:**

- Vector result
- Not commutative: $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$

✅ **Applications:**

- Torque: $\vec{\tau} = \vec{r} \times \vec{F}$
- Angular momentum: $\vec{L} = \vec{r} \times \vec{p}$
- Magnetic force: $\vec{F} = q\vec{v} \times \vec{B}$

📘 4. Vector Components in 2D and 3D

✅ In 2D:

$$\vec{A} = A_x\hat{i} + A_y\hat{j} \quad \text{with} \quad A_x = A\cos\theta, \quad A_y = A\sin\theta$$

✅ In 3D:

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

✅ Magnitude:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

📘 5. Applications in Physics

Vector Quantity	Example Equation	Context
Force	$\vec{F} = m\vec{a}$	Newton's 2nd law
Velocity	$\vec{v} = \frac{d\vec{r}}{dt}$	Motion
Acceleration	$\vec{a} = \frac{d\vec{v}}{dt}$	Kinematics
Momentum	$\vec{p} = m\vec{v}$	Collisions
Torque	$\vec{\tau} = \vec{r} \times \vec{F}$	Rotation
Electric field	$\vec{E} = \frac{\vec{F}}{q}$	Electrostatics
Magnetic force	$\vec{F} = q\vec{v} \times \vec{B}$	Electromagnetism

Calculus in Physics

📘 1. Differentiation

✅ Definition:

If $y = f(x)$, the derivative is:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

It gives the **instantaneous rate of change** of y with respect to x.

✅ Applications in Physics:

Quantity	Expression	Meaning
Velocity	$v(t) = \frac{dx}{dt}$	Rate of change of position
Acceleration	$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	Rate of change of velocity
Slope of a graph	$\frac{dy}{dx}$	Used to find tangents
Force from potential energy	$F(x) = -\frac{dU}{dx}$	Conservative forces
Current from charge	$I = \frac{dq}{dt}$	Rate of flow of charge

Sum/Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Chain Rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \quad \text{and} \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \text{and} \quad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \quad \text{and} \quad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \quad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

Derivatives of Exponential and Logarithmic Functions

$$\frac{d}{dx}(e^x) = e^x \quad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x, \quad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x \quad \frac{d}{dx}(\coth x) = -\text{csch}^2 x$$